Quasi-classical $\bar{\partial}$-method: Generating equations for dispersionless integrable hierarchies

L.V. Bogdanov*, B.G. Konopelchenko† and L. Martinez Alonso‡

Abstract

The quasi-classical $\bar{\partial}$-dressing method is used to derive compact generating equations for dispersionless hierarchies. Dispersionless Kadomtsev-Petviashvili (KP) and two-dimensional Toda lattice (2DTL) hierarchies are considered as illustrative examples.

1 Introduction

Dispersionless integrable equations and corresponding hierarchies have attracted a considerable interest during the last two decades. Their study is of great importance since dispersionless integrable hierarchies constitute an essential part of the general theory of integrable systems (see e.g. [1]-[13]) and they arise in the analysis of many problems in physics, mathematics and applied mathematics (see e.g. [14]-[21]).

Recently it was shown that dispersionless integrable hierarchies are amenable to the quasi-classical $\bar{\partial}$-dressing method [22, 23, 24].This approach provides us with a simple and elegant method of constructing and solving dispersionless integrable hierarchies. Moreover, it establishes a connection between these hierarchies and the theory of quasi-conformal mappings on the plane [23].

In the present paper we use the quasi-classical $\bar{\partial}$-dressing method in order to derive compact generating equations for dispersionless integrable hierarchies. These equations, in particular, imply the existence of τ-functions and, in a very simple way, provide us with the corresponding dispersionless addition formulae (Hirota equations). The dispersionless KP (dKP) and 2DTL (d2DTL) hierarchies are considered as illustrative examples of the general approach.

*L.D. Landau ITP, Kosygin str. 2, Moscow 117940, Russia

†Dipartimento di Fisica dell' Università and Sezione INFN, 73100 Lecce, Italy

‡Departamento de Fisica Teorica II Universidad Complutense, Madrid, Spain

2 Quasi-classical $\bar{\partial}$-method

The quasi-classical $\bar{\partial}$-dressing method proposed in [22, 23, 24] is based on the quasi-classical $\bar{\partial}$-problem

$$S_{\bar{z}} = W(z, \bar{z}, S_z), \tag{2.1}$$

where $z \in \mathbb{C}$, bar means complex conjugation,

$$S_z = \frac{\partial S(z, \bar{z})}{\partial z}$$

and W (quasi-classical $\bar{\partial}$-data) is an analytic function of S_z. We are looking for solutions of equation (2.1) in the form

$$S = S_0 + \widetilde{S},$$

where $S_{0\bar{z}} = 0$ for $z \in G$ (G is certain domain in $\mathbb{C}$) and $\widetilde{S}_{\bar{z}} = 0$ for $z \in \mathbb{C} \backslash G$. Parameterizing a function S_0 which is analytic in G by certain (infinite) set of parameters (times), i.e., $S_0 = S_0(z; t_1, t_2, \ldots)$, one thus has solutions of equation (2.1) depending on these parameters: $S = S(z, \bar{z}; t_1, t_2, \ldots)$. A standard way to describe such a dependence is provided by partial differential equations for S with $t_1, t_2, \ldots$ as independent variables.

To derive these equations, one notices that equation (2.1) implies that

$$\left(\frac{\partial S}{\partial t_i}\right)_{\bar{z}} = W'(z, \bar{z}; S_z) \left(\frac{\partial S}{\partial t_i}\right)_z, \quad i = 1, 2, 3, \ldots, \tag{2.2}$$

where

$$W'(z, \bar{z}; \xi) = \frac{\partial W(z, \bar{z}; \xi)}{\partial \xi}.$$

The Beltrami equation on the plane

$$f_{\bar{z}} = \mu f_z$$

has a number of remarkable properties (see e.g. [25]). Two of them are crucial for the quasi-classical $\bar{\partial}$-method. They are:

1. if f_1 and f_2 are solutions of the Beltrami equation then $F(f_1, f_2)$ where F is an arbitrary differentiable function is a solution too;

2. if f is the solution of the Beltrami equation which is bounded in $\mathbb{C}$ and $f \to 0$ as $z \to z_0$ ($z_0 \in \mathbb{C} \cup \infty$), then (under certain mild conditions) $f = 0$.

These two properties imply that, first, $F(\frac{\partial S}{\partial t_1}, \frac{\partial S}{\partial t_2}, \ldots)$, where F is an arbitrary differentiable function, obeys the Beltrami equation $F_{\bar{z}} = \mu F_z$ and

second, if F is bounded as a function of z and $F \to 0$ as $z \to z_0$, then $F \equiv 0$. Thus such functions provide us with equations [24]

$$F_i \left(\frac{\partial S}{\partial t_1}, \frac{\partial S}{\partial t_2}, \ldots \right) = 0, \quad i = 1, 2, \ldots \tag{2.3}$$

These equations are our desired differential equations for dispersionless hierarchies. The form of equations (2.3) is completely determined by the choice of the domain G and parameterization of $S_0(z; t_1, t_2, \ldots)$.

Thus, the quasi-classical $\bar{\partial}$-dressing method consists basically in rather elementary operations of complex analysis applied to equation (2.1).

3 dKP hierarchy

In the paper [24] the quasi-classical $\bar{\partial}$-dressing method has been used to derive dKP, dmKP and d2DTL hierarchies in their usual formulations. Here we present the derivation of generating equations which encode different forms of dispersionless hierarchies, and thus play a central role in their theory.

We begin with the dKP hierarchy. In this case the domain G is the unit disk ($|z| \le 1$), $W = 0$ for $|z| > 1$, and $S_0 = \sum_{n=1}^{\infty} z^n t_n$ [24]. We also require $\widetilde{S} \sim \widetilde{S}_1 z^{-1} + \widetilde{S}_2 z^{-2} + \ldots$ as $z \to \infty$. The quantity $p = \frac{\partial S}{\partial t_1}$ is a basic homeomorphism [23] and

$$p = z + \frac{1}{z} \frac{\partial \widetilde{S}_1}{\partial t_1} + \ldots$$

as $z \to \infty$. Let us introduce the well-known operator

$$D(z) = \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{z^n} \frac{\partial}{\partial t_n}.$$

In virtue of the properties of equation (2.1) and Beltrami equation (2.2) the function $D(z_1)S(z)$ ($z \in \mathbb{C}$, $z_1 \in \mathbb{C} \setminus G$) and consequently $\exp(-D(z_1)S(z))$ obey equation (2.2). Since $D(z_1)S_0(z) = -\log(1 - \frac{z}{z_1})$, one has

$$\exp(-D(z_1)S(z)) = \left(1 - \frac{z}{z_1}\right) \exp(-D(z_1)\widetilde{S}(z)). \tag{3.1}$$

Thus, $\exp(-D(z_1)S(z)) \sim -\frac{1}{z_1} z + O(1)$ as $z \to \infty$. Using these properties of the functions $p(z)$ and $\exp(-D(z_1)S(z))$, one constructs an equation of the form (2.3). Namely,

$$p(z) + z_1 \exp(-D(z_1)S(z)) - (z_1 + D(z_1)\widetilde{S}_1) = 0. \tag{3.2}$$

On the other hand, evaluating the l.h.s. of (3.2) at $z = z_1$ and using (3.1), one gets its equivalent form

$$p(z) - p(z_1) + z_1 \exp(-D(z_1)S(z)) = 0, \quad z \in \mathbb{C}, \quad z_1 \in \mathbb{C} \setminus G. \tag{3.3}$$

It is easy to check this equation directly. Indeed, the l.h.s. is a bounded solution of Beltrami equation (2.2) having zero at $z = z_1$. Then the properties of Beltrami equation imply that it is zero identically.

Equation (3.3) occupies the central place in the theory of dKP hierarchy. This fact has been already understood in [5, 7].However, we would like to emphasize that in equation (3.3) z is an arbitrary point in $\mathbb{C}$, while the corresponding equation in [5, 7] is valid for $z \in \mathbb{C} \setminus G$ only. The fact that $z \in \mathbb{C}$ in equation (3.3) will be essential in some of further constructions.

An immediate consequence of equation (3.3) together with (3.1) is that for $z, z_1 \in \mathbb{C} \setminus G$

$$D(z_1)\widetilde{S}(z, \mathbf{t}) = D(z)\widetilde{S}(z_1, \mathbf{t}).$$

So, there exists a function $F(\mathbf{t})$ such that

$$\widetilde{S}(z, \mathbf{t}) = -D(z)F(\mathbf{t}), \quad z \in \mathbb{C} \setminus G.$$

Thus, one has (see also [5, 7, 12])

$$p(z_1) - p(z_2) = (z_1 - z_2)\exp(D(z_1)D(z_2)F), \quad z_1, z_2 \in \mathbb{C} \setminus G. \tag{3.4}$$

Considering equation (3.4) for pairs of parameters (z_1, z_2), (z_2, z_3), (z_3, z_1) and summing them up, one gets

$$\begin{aligned}(z_1 - z_2)e^{D(z_1)D(z_2)F} + (z_2 - z_3)e^{D(z_2)D(z_3)F} + (z_3 - z_1)e^{D(z_3)D(z_1)F} \\ = 0, \\ z_1, z_2, z_3 \in \mathbb{C} \setminus G,\end{aligned} \tag{3.5}$$

that is the dispersionless Fay identity [7, 11]. Thus $F = \log \tau_{\mathrm{dKP}}$, where τ_{dKP} is the τ-function of the dKP hierarchy [7, 11].

Equation (3.3) generates the hierarchy of Hamilton-Jacobi type equations for the dKP hierarchy. Indeed, rewriting (3.3) as

$$\frac{\partial S(z)}{\partial t_1} - \frac{\partial S(z_1)}{\partial t_1} + z_1 e^{-D(z_1)S(z)} = 0, \tag{3.6}$$

and expanding its l.h.s. as $z_1 \to \infty$, one gets the equations

$$-\frac{\partial \widetilde{S}_1(t)}{\partial t_1} - \frac{1}{2}\frac{\partial S(z)}{\partial t_2} + \frac{1}{2}\left(\frac{\partial S(z)}{\partial t_1}\right)^2 = 0, \tag{3.7}$$

$$-\frac{\partial \widetilde{S}_2(t)}{\partial t_1} - \frac{1}{6}\frac{\partial S(z)}{\partial t_3} + \frac{1}{6}\frac{\partial S(z)}{\partial t_1}\frac{\partial S(z)}{\partial t_2} - \frac{1}{6}\left(\frac{\partial S(z)}{\partial t_1}\right)^3 = 0$$

and so on, that represents one of the equivalent forms of the known Hamilton-Jacobi type equations for dKP hierarchy.

Equation (3.3) allows us to derive an equation which generates the hierarchy of equations for $S(z,\bar{z};\mathbf{t})$ only. In fact, acting by $D(z_k)$ on equation (3.2) (with substitution $z_1 \to z_i$), one gets

$$D(z_k)p(z) - D(z_k)D(z_i)\widetilde{S}_1 + z_i D(z_k) e^{-D(z_k)S(z)} = 0, \qquad z \in \mathbb{C}, \quad z_i, z_k \in \mathbb{C} \setminus G. \tag{3.8}$$

Taking equation (3.8) with all possible pairs z_i, z_k $(i,k = 1,2,3)$ and summing up, one obtains the equation

$$\sum_{i,j,k} \epsilon_{ijk} z_i D(z_k) \left(e^{-D(z_i)S(z,\bar{z};\mathbf{t})} \right) = 0, \qquad z \in \mathbb{C}, \quad z_1, z_2, z_3 \in \mathbb{C} \setminus G, \tag{3.9}$$

where ϵ_{ijk} is the totally antisymmetric tensor (with $\epsilon_{123} = 1$) and summation is performed over all $i,j,k \in 1,2,3$. Considering equation (3.9) for $z_1, z_2, z_3 \to \infty$ and collecting the terms of different orders in z_1^{-1}, z_2^{-1} and z_3^{-1}, one gets an infinite hierarchy of differential equations for $S(z,\bar{z};\mathbf{t})$. In the lowest nontrivial order $(z_3^{-1}z_1^{-2},\ z_3^{-1}z_2^{-2},\ \ldots)$ one has

$$\frac{\partial^2 S(z,\bar{z};\mathbf{t})}{\partial t_1 \partial t_3} - \frac{3}{4}\frac{\partial^2 S}{\partial t_2^2} - \frac{3}{2}\frac{\partial^2 S}{\partial t_1^2}\left(\frac{\partial S}{\partial t_2} - \left(\frac{\partial S}{\partial t_1} \right)^2 \right) = 0, \tag{3.10}$$

which, of course, is also a consequence of equations (3.7).

Further, expanding the l.h.s. of (3.9) in the Taylor series in z^{-1} as $z \to \infty$, one gets (in the order z^{-1})

$$\sum_{i,j,k} \epsilon_{ijk} z_i D(z_k) \left(z_i D(z_i)\widetilde{S}_1 + \frac{1}{2}\left(D(z_i)\widetilde{S}_1 \right)^2 \right) = 0. \tag{3.11}$$

This equation is the generating equation of the whole dKP hierarchy (for $u(\mathbf{t}) = -2\frac{\partial \widetilde{S}_1}{\partial t_1}$). it is a simple check that the expansion of (3.11) at $z_1, z_2, z_3 \to \infty$ gives rise to the dKP equation

$$u_{t_1 t_3} = \frac{3}{2}(u u_{t_1})_{t_1} + \frac{3}{4} u_{t_2 t_2}$$

and higher equations.

Thus, equation (3.9) is one of the fundamental equations for the dKP hierarchy. Note that since $S(z,\bar{z};\mathbf{t})$ defines a quasi-conformal mapping of the

domain G [23], equation (3.9) is also the central equation for the integrable deformations of quasi-conformal mappings.

Finally, we present one more equation associated with the dKP hierarchy. Using (3.3), one gets

$$e^{D(z_1)(S(z_2)-S(z))} + e^{D(z_2)(S(z_1)-S(z))} = 1, \qquad z \in \mathbb{C}, \quad z_1, z_2 \in \mathbb{C} \setminus G. \tag{3.12}$$

Consequently,

$$e^{D(z_i)(S_j(\mathbf{t})-S_k(\mathbf{t}))} + e^{D(z_j)(S_i(\mathbf{t})-S_k(\mathbf{t}))} = 1, \tag{3.13}$$

where $i, j, k \in 1, 2, 3,\ i \neq j \neq k \neq 1,\ S_i(\mathbf{t}) = S(z_i, \bar{z}_i; \mathbf{t}),\ z_1, z_2, z_3 \in \mathbb{C} \setminus G$. This system of three equations for three functions $S_1(\mathbf{t}), S_2(\mathbf{t}), S_3(\mathbf{t})$ can be considered as a dispersionless version of the Darboux system.

4 2DTL hierarchy

For the dispersionless 2DTL hierarchy the domain G is an annulus $a < |z| < b$, where a, b ($a, b \in \mathbb{R},\ a, b > 0;\ b > a$) are arbitrary. To set the quasi-classical $\bar{\partial}$-problem (2.1) correctly, in general we do not need to require analyticity of the function S_0 in G, it is enough to have analyticity of its derivative S_{0z}. A generic function S_0 with S_{0z} analytic in G can be represented as

$$S_0 = t \log z + \sum_{n=1}^{\infty} z^n x_n + \sum_{n=1}^{\infty} z^{-n} y_n,$$

where t, x_n, y_n are free parameters [24]. We assume that $\widetilde{S}(z) \sim \sum_{n=1}^{\infty} \frac{S_n}{z^n}$ as $z \to \infty$ and denote $\widetilde{S}(0) = \phi$, $G_+ = \{z, |z| > b\}$, $G_- = \{z, |z| < a\}$. The functions $p_+ = \frac{\partial S}{\partial x_1}$ and $p_- = \frac{\partial S}{\partial y_1}$ have pole singularities while $p = \frac{\partial S}{\partial t}$ has a logarithmic singularity. The functions $\exp(-D_+(z_1)S(z))$, $\exp(-D_-(z_2)S(z))$, $\exp(DS(z))$, where $z \in \mathbb{C}$, $z_1 \in G_+$, $z_2 \in G_-$ and

$$D_+(z) = \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{z^n} \frac{\partial}{\partial x_n}, \quad D_-(z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n \frac{\partial}{\partial y_n}, \quad D = \frac{\partial}{\partial t},$$

are solutions of the Beltrami equation and have pole singularities too.

Using these functions, one gets the following equations

$$p_+(z) - p_+(z_1) + z_1 e^{-D_+(z_1)S(z)} = 0, \quad z \in \mathbb{C}, z_1 \in G_+, \tag{4.1}$$

$$p_-(z) - p_-(z_2) + \frac{1}{z_2} e^{D_-(z_2)(\phi - S(z))} = 0, \quad z \in \mathbb{C}, z_2 \in G_-, \tag{4.2}$$

$$p_+(z) = e^{p(z) - D\widetilde{S}_1}, \quad z \in \mathbb{C}, \tag{4.3}$$

$$p_-(z) = e^{D\phi - p(z)}, \quad z \in \mathbb{C}, \tag{4.4}$$

Equation (4.1) coincides with (3.3) and gives rise to the dKP hierarchy in variables x_n. In particular, one has

$$\widetilde{S}(z_1) = -D_+(z_1)F_+, \quad z_1 \in G_+$$

as well as the corresponding dKP addition formula and generating equations.

Equation (4.2) encodes the dmKP hierarchy. It implies that

$$\widetilde{S}(z_2) = \phi - D_-(z_2)F_-, \quad z_2 \in G_-,$$

the function F_- obeys the dmKP addition formula which coincides with (3.5) under the substitution $F \to F_-$, $D \to D_-$ and $z \to z^{-1}$. It is a simple exercise to derive the corresponding generating equations.

The formulae (4.3) and (4.4) allow us to connect these two pieces into the d2DTL hierarchy. Using (4.3), one rewrites (4.4) as

$$e^{DS(z)} - e^{DS(z_1)} + z_1 e^{-D_+(z_1)S(z)} = 0, \quad z \in \mathbb{C}, z_1 \in G_+, \tag{4.5}$$

while using the relation $p_-(z_2) = z_2^{-1} \exp(D_-(z_2)\phi)$ and (4.4), one transforms (4.2) into

$$e^{DS(z)} \left(1 - e^{D_-(z_2)S(z)}\right) = z_2 e^{(D - D_-(z_2))\phi}, \quad z \in \mathbb{C}, z_2 \in G_-. \tag{4.6}$$

Evaluating the l.h.s. of equation (4.5) at $z = 0$, one gets

$$\exp(DS(z_1)) = z_1 \exp(-D_+(z_1)\phi).$$

It implies that $\phi = DF_+$. On the other hand, evaluation of relation (4.6) around $z_2 = 0$ gives

$$\frac{\partial S(z)}{\partial y_1} \exp(DS(z)) = \exp(D\phi).$$

Expanding the l.h.s. of this relation around $z = 0$, one gets

$$\frac{\partial^2 F_-}{\partial y_1 \partial t} = \frac{\partial \phi}{\partial y_1} = \frac{\partial^2 F_+}{\partial y_1 \partial t}$$

and so on. Thus, $F_+ = F_- = F$. Since $\widetilde{S}(z_1) = -D_+(z_1)F$ $(z_1 \in G_+)$, $\widetilde{S}(z_2) = D\phi - D_-(z_2)F$ $(z_2 \in G_-)$, one obtains from (4.5) and (4.6) the following known addition formulae (dispersionless Hirota equations) for the d2DTL hierarchy (see [11, 20, 26]):

$$\widetilde{z}_1 e^{-DD_+(\widetilde{z}_1)F} - z_1 e^{-DD_+(z_1)F} + (z_1 - \widetilde{z}_1) e^{D_+(z_1)D_+(\widetilde{z}_1)F} = 0, \quad z_1, \widetilde{z}_1 \in G_+, \tag{4.7}$$

$$1 + \left(\frac{z_2}{z_1} - 1\right) e^{D_-(z_2)D_+(z_1)F} - \frac{z_2}{z_1} e^{(D_+(z_1)+D-D_-(z_2))DF} = 0, \quad z_1 \in G_+, z_2 \in G_- \tag{4.8}$$

plus the dKP addition formula (3.5). Note that evaluating the l.h.s. of equation (4.5) as $z \to \infty$, one also gets

$$\frac{\partial^2 F}{\partial x_1 \partial y_1} = 1 - e^{D^2 F}. \tag{4.9}$$

Equations (4.5) and (4.6) are also the generating equations for the hierarchy of the Hamilton-Jacobi type equations. Indeed, expansion of the l.h.s. of equation (4.5) for $z_1 \to \infty$ gives

$$\frac{\partial S(z)}{\partial x_1} - e^{\frac{\partial S}{\partial t}} + \frac{\partial \widetilde{S}_1}{\partial t} = 0, \tag{4.10}$$

$$\frac{\partial S}{\partial x_2} - \left(\frac{\partial S}{\partial x_1}\right)^2 + \frac{\partial \widetilde{S}_2}{\partial t} + \frac{1}{2}\left(\frac{\partial \widetilde{S}_1}{\partial t}\right)^2 = 0$$

and so on, while the expansion of the equations around $z_2 = 0$ provides us with the equations

$$\frac{\partial S}{\partial y_1} - e^{\frac{\partial \phi}{\partial t} - \frac{\partial S}{\partial t}} = 0, \tag{4.11}$$

$$\frac{\partial S}{\partial y_2} - \left(\frac{\partial S}{\partial y_1}\right)^2 + \frac{\partial \phi}{\partial y_1} e^{\frac{\partial \phi}{\partial t} - \frac{\partial S}{\partial t}} = 0$$

plus higher equations which are equivalent to those found in [24].

Now let us derive generating equations for $S(z, \bar{z}; t, \mathbf{x}, \mathbf{y})$ and $\phi(t, \mathbf{x}, \mathbf{y})$. First, equation (4.5) implies

$$-D_+(z_1)\phi = \log\left(\frac{1}{z_1} e^{DS(z)} + e^{-D_+(z_1)S(z)}\right), \quad z_1 \in G_+, z \in \mathbb{C}, \tag{4.12}$$

while from (4.6) one obtains

$$(D - D_-(z_2))\phi = \log\left(\frac{1}{z_2}e^{DS(z)} - \frac{1}{z_2}e^{(D-D_-(z_2))S(z)}\right), \quad z_2 \in G_-, z \in \mathbb{C}. \tag{4.13}$$

Eliminating ϕ from these equations, one gets

$$\begin{aligned}(D - D_-(z_2))\log\left(\frac{1}{z_1}e^{DS(z)} + e^{-D_+(z_1)S(z)}\right)& \\ +D_+(z_1)\log\left(e^{DS(z)} - e^{(D-D_-(z_2))S(z)}\right) &= 0,\end{aligned} \tag{4.14}$$

where $z \in \mathbb{C}$, $z_1 \in G_+$, $z_2 \in G_-$. Considering the limit $z_1 \to \infty$, $z_2 \to 0$, one obtains in the lowest order the equation

$$\frac{\partial^2 S}{\partial x_1 \partial y_1} + \frac{\partial S}{\partial y_1}\frac{\partial}{\partial t}\left(e^{\frac{\partial S}{\partial t}}\right) = 0, \tag{4.15}$$

which has been already found in [24]. Higher terms in the expansion give rise to higher equations for $S(z, \bar{z}; t, \mathbf{x}, \mathbf{y})$.

To derive generating equations for ϕ let us rewrite equation (4.8) as

$$D_-(z_2)D_+(z_1)F = \log\left(\frac{z_1}{z_1 - z_2}\left(1 - \frac{z_2}{z_1}e^{(D_+(z_1)+D-D_-(z_2))DF}\right)\right).$$

Applying D to this equation and using the relation $DF = \phi$, one gets

$$D_-(z_2)D_+(z_1)\phi = D\log\left(\frac{z_1}{z_1 - z_2}\left(1 - \frac{z_2}{z_1}e^{(D_+(z_1)+D-D_-(z_2))\phi}\right)\right), \tag{4.16}$$
$$z_1 \in G_+, z_2 \in G_-.$$

An expansion of both sides of equation (4.16) as $z_1 \to \infty$ and $z_2 \to 0$ in the lowest nontrivial order (z_1^{-1}, z_2)gives the d2DTL equation (see e.g. [10, 11])

$$\frac{\partial^2 \phi}{\partial x_1 \partial y_1} + \frac{\partial}{\partial t}\left(e^{\frac{\partial \phi}{\partial t}}\right) = 0, \tag{4.17}$$

while considering higher order terms, one gets higher equations from the d2DTL hierarchy.

The generating equations (3.9) and (4.14) represent also the compact forms of integrable deformations of quasi-conformal mappings $S(z, \bar{z}; t, \mathbf{x}, \mathbf{y})$ of the unit disk and an annulus respectively. Note that one can derive the same formulae using the finite number of 'logarithmic' times ξ_i defined by

$$S_0(z, \xi) = \sum_{i=1}^{3} \xi_i \log(z - z_i)$$

instead of infinite sets of times t_n or x_n, y_n.

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